Proceedings of IDETC/CIE 2005
ASME 2005 International Design Engineering Technical Conferences & Computers and Information in Engineering Conference
September 24-28, 2005, Long Beach, California, USA

DETC2005-84773

COMPUTING ADMISSIBLE TRANSFORMATION VOLUME

Xiaoping Qian, Assistant Professor
Department of Mechanical, Materials and Aerospace Engineering
Illinois Institute of Technology
Chicago, IL 60616
qian@iit.edu

Pinghai Yang, Graduate Research Assistant
Department of Mechanical, Materials and Aerospace Engineering
Illinois Institute of Technology
Chicago, IL 60616
yangpin@iit.edu

ABSTRACT
The ability to quantify part dimensional quality with respect to design specifications is of fundamental importance in product design and manufacturing. Our earlier work has proposed the use of admissible transformation volume as a part dimensional quality metric. That is, part quality is quantified based on how much an as-manufactured part shape can move while still remaining within a tolerance zone. A transformation is admissible if upon such a transformation a manufactured part shape falls within the design tolerance zone. A collection of such transformations in the transformation space forms an *admissible transformation volume* (ATV). In this paper, we present two properties of ATV: transformation invariant and decomposability. We then describe algorithms for computing ATV and how ATV properties facilitate complex tolerance check and reveal new insight on part producibility.

INTRODUCTION
The ability to quantify part dimensional quality with respect to design specifications is of fundamental importance in product design and manufacturing. Many industries, such as aerospace, automobile, and die and mold industries, are striving for the design and manufacture of technically advanced products that deliver superior performance with longer life. Many of these products incorporate components designed with tighter tolerances and manufactured with improved dimensional control. At the same time these products often consist of geometrically complicated features under complex tolerance schemes that serve distinct functions in various engineering applications. The ability to design these products with tighter tolerances and to produce them with improved dimensional control in a cost effective manner is essential for these business to thrive in a competitive environment.

The recent advancement of 3D optical scanning systems and the rapid proliferation of coordinate measuring machines (CMMs) have made part coordinate data ubiquitous and readily available. Such readily available coordinate data makes it possible to improve part producibility in a cost-effective manner by quantitatively analyzing actual part dimensional quality from the measured part coordinate data. The ability to extract dimensional quality data from part coordinate data and to quantify the influence of tolerance specifications and manufacturing error variations over the part geometric dimensioning and tolerancing (GD&T) conformance is essential for part producibility improvement. Existing dimensional quality analysis methods are based on either the deviation between as-measured part data and the nominal model or the minimal tolerance zone of the measured data. These methods are either not conformal to ANSI Y14.5M standard [1] or not directly applicable to complex tolerance such as non-uniform tolerance and composite tolerance. Some of these methods are dedicated to particular classes of tolerances and are computationally undesirable. Furthermore, **these methods cannot effectively evaluate part dimensional quality when multiple tolerance requirements need to be simultaneously met**.

Figure 1 presents an example in which a nominal geometry of an airfoil cross-section is subject to uniform profile tolerance (Figure 1.b) and non-uniform profile tolerance (Figure 1.c). Conventional approaches to part dimensional quality gauging characterize a manufactured part based on the minimal tolerance zone, a theoretical (minimal maximum) measure of deviation from nominal geometry. However, when the section profile is under non-uniform tolerance, *e.g.* the leading edge needs to be under tighter dimensional control, the minimal tolerance for the contour becomes ambiguous or “*conditional*” [5]. Since the minimal surface profile tolerances in the looser tolerance zone and in the tighter tolerance zone (in the leading edge area) are different and in fact they are inter-dependent. Thus, this leads to difficulty in characterizing the profile quality based on minimal tolerance zone and subsequently further difficulty in assigning design tolerance and in improving manufacturing processes.

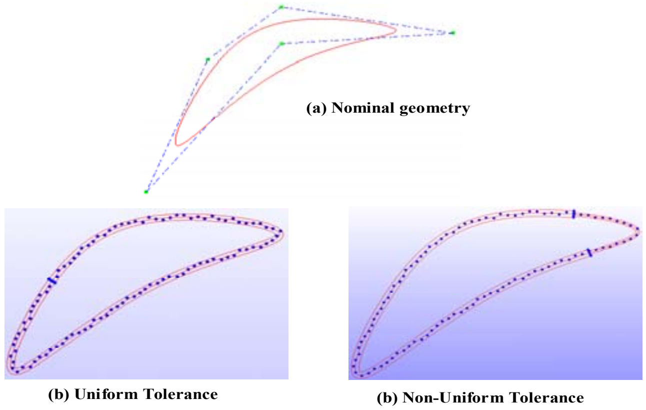


Figure 1: Part dimensional quality in a non-uniform tolerance zone

We have proposed a new approach to part dimensional quality gauging, based on a novel concept *admissible transformation volume [19]*. In this approach, part quality is quantified based on how much an as-manufactured part shape can move while still remaining within a tolerance zone. A transformation is admissible if upon such a transformation a manufactured part shape falls within the design tolerance zone. A collection of such transformations in the transformation space forms an admissible transformation volume (ATV). This approach measures part quality through the ATV. Its advantages over current dimensional quality analysis methods are its conformance to ANSI Y14.5M standard and applicability to a variety of geometric dimensioning and tolerancing classes including non-uniform tolerance. This metric provides for the first time a spatial metric for part dimensional quality, enabling the visualization of part quality.

In this paper, we present two properties of ATV: transformation invariant and decomposability. In addition, we present a set of algorithms for computing ATV and a set-theoretic approach for representing and computing ATV for complex tolerance based on the decomposable computing characteristics. We demonstrate how ATV properties facilitate complex tolerance check and reveal new insight on part producibility.

In the remainder of this paper, we review prior work on part dimensional quality analysis in Section 2. We present formal definitions and properties of ATV in Section 3. Algorithms for computing ATV are described in Section 4 and the experimental implementation in Section 5. This paper is concluded in Section 6.

REVIEW OF PRIOR WORK

Part dimensional quality analysis requires the comparison of measured part coordinate data with respect to part GD&T specifications. GD&T is an important technology in product design and manufacturing. Through GD&T, design intent can be represented, part quality can be analyzed, part interoperability from various manufacturing processes and different vendors can be ensured, and manufacturing cost can be reduced.

Functional and assembly requirements on the manufactured parts are represented as tolerance zones to which the surface of a part must conform. These geometric tolerances are defined in the ASME Y14.5M-1994 geometric dimensioning and tolerancing standard [1]. Based on the standard, tolerances are to be evaluated from envelopes of two ideal features with minimum separation distance within which the entire surface of the manufactured part must lie.

To analyze whether a manufactured part meets design tolerance specifications from a set of part coordinate data, one needs a proper representation of tolerance and an appropriate methodology to compare measured coordinate data with the tolerance. Such comparisons are used not only to determine the qualification of the manufactured part, but also to extract quantitative part quality information that can be fed back for process modification as well as design change for producibility improvement.

In this section, we briefly review GD&T theories as well as methods to construct tolerance zones. We then present past and current methods on part dimensional quality analysis.

GEOMETRIC DIMENSIONING AND TOLERANCING THEORIES

Manufactured parts have deviations from the nominal shape. To describe and preserve the functional requirements of design, geometric variations are specified in tolerance zones. Pasupathy *et al* gave a comprehensive review of various existing tolerance zone construction methods in [17].

Offset zone models modeled as Boolean subtraction of maximal and minimal object volumes have been explored by Requicha [20] and Roy [21]. Turner developed indirect parameterization methods for modeling tolerance zones [24]. A Technologically and Topologically Related Surfaces (TTRS) method was developed by Clement [6], where they used group theory and displacement torsors to combine the surfaces into 28 different geometric relationships. Shah and Zhang developed a graph-based model for geometric tolerancing by separating linear variations from angular variations based on degrees of freedom for points, lines, and planes [23].

Recently Davidson and Shah proposed a new mathematical model, Tolerance-Map, a hypothetical volume of points that corresponds to all possible locations and variations of a segment of a plane which can arise from tolerances in size, form, and orientation [8]. A GD&T global model for computerizing GD&T representation was reported in [25].

In this paper, we focus on developing a measure of part dimensional quality in its conformance to GD&T specifications. We assume the tolerance zone Z is represented as a parametric function of nominal geometry S. That is, for a given surface point in its parametric representation S(u,v), we can compute the tolerance zone Z(u,v) at that point. The methodology and the metric developed in this paper are applicable to other tolerance representations as well.

DIMENSIONAL QUALITY ANALYSIS FROM MEASUREMENT DATA

Dimensional quality analysis of measured coordinate points serves two important purposes: 1) to check whether a manufactured part meets design GD&T specifications (*qualitative GD& T conformance check*), 2) to characterize manufacturing process capability and examine how much space remains for a manufactured part shape to stay within a tolerance zone *(quantitative characterization of part dimensional quality and process capability*). The first question concerns whether a manufactured part meets design tolerance specification. The second question concerns whether the manufactured part just fits into a tolerance zone or if there is ample space remaining. This quantitative characterization of part quality is critical for improving part producibility. Part dimensional quality is often evaluated based on the minimum tolerance zone computed from the measured dimensional data. This evaluation is often based on numerical fitting algorithms that transform the measured coordinate data into the nominal geometry's coordinate system to minimize the deviation between the nominal shape and the inspection point set. The fitting algorithms can be largely divided into two types: least-squares fit and mini-max fit. Refer to Feng [10] for a detailed review of various fitting algorithms.

An alternative to the numerical fitting algorithms is a combinatorial search for particular points that control and govern the minimum tolerance zone. In addition, manual fitting is still employed for some complex and high precision part inspection.

We now review these dimensional quality analysis methods and explain why our proposed approach is advantageous for part dimensional quality analysis.

NUMERICAL FITTING BASED TOLERANCE EVALUATION

The numerical fitting based methods, including least-squares fit, mini-max fit, and zone fit, are relatively easy to implement and are applicable to a variety of GD&T classes. In general, these methods are fast but subject to potential errors due to numerical approximations and the lack of true global optimization algorithms.

Total least-square fitting calculates deviation for all the inspection points and then sums its deviations. For example, Menq used this method for surface profile inspection [15]. This method minimizes the overall root-mean-square error, but may lead to a larger maximum deviation. Therefore total least-squares fitting could over-estimate the tolerance values, which would unnecessarily disqualify many otherwise qualified parts.

To resolve the inconsistency between design intent of tolerance specifications and the least-squares fit, an alternative fitting method, minim-max fit, has been developed. Minimum tolerance is computed based on the maximum deviation between the nominal geometry and the measured point set. For example, Murthy used a Monte Carlo simulation algorithm to determine the minimal tolerance zone for form tolerance [16]. Lai modified a genetic algorithm for calculating the minimum-zone for cylindricity [13]. Mini-max fit is useful for estimating tolerances such as roundness, cylindricity and flatness. It is not directly applicable to shapes with non-uniform tolerance bands, or with asymmetric tolerance bands. Mini-max fitting minimizes the largest deviation error but it may lead to alignment with larger overall root-mean-square error. It is also computationally undesirable since the first derivative of the objective function may not be continuous.

Recognizing the deficiencies of the two types of fitting algorithms, Choi and Kurfess developed a zone-fitting algorithm [4], in which a quasi-Newton method is used to numerically seek a rigid body transformation placing the inspection points inside the tolerance zone. A minimum tolerance zone is an effective metric for part quality characterization, it becomes ineffective when multiple tolerance zones are involved. To address multiple tolerances, a conditional tolerance zone concept is proposed in [5]. The minimum tolerance zone for a tolerance feature is computed while holding a constant tolerance zone on the other tolerance features. This would unfortunately lead to multiple minimum tolerance zone values for a given tolerance feature. It would also involve combinatorial evaluation of minimal tolerance zones for multiple tolerance features.

The admissible transformation is similar to the zone fitting method in that both approaches compare inspection points with a tolerance zone. However, we explicitly quantify the amount of admissible transformation (ATV) in the transformation space and examine quantitatively the ATV change due to design tolerance specifications change and manufacturing error variation. As such, ATV is applicable to both single and multiple tolerance zone specifications.

COMBINATORIAL MINIMUM ZONE COMPUTING

Various geometric approaches have also been explored to calculate the minimal tolerance zone. In these approaches, points that control the minimum tolerance zone are explicitly identified. Huang used a method called control line rotation scheme to identify points to calculate minimum-zone straightness [10]. Damodarasamy used a normal plane method and simplex search for calculating the minimum zone for flatness [7]. Roy and Zhang constructed the nearest and farthest Voronoi diagrams of a data set for circularity evaluation [22]. The minimum tolerance zone issue has also been formed as an annulus placement issue in computational geometry [2].

Due to the combinatorial nature of these algorithms, they are computationally expensive and are dedicated to particular types of tolerances and not applicable for general classes of tolerances.

Besides the above automatic fitting methods, another method that is often used in checking part dimensional quality is through the use of manual fit. In this method, a blueprint drawing with the tolerance zone is magnified and printed on a Mylar or plastic paper. The actual part profile is then superimposed against the blue print. The advantage of this approach is that it conforms to design intent of tolerance specifications. However, despite its wide usage in high precision and complex profile part inspection, this method also has many disadvantages. It is subjective, not repeatable, and relies on operators' judgment. More importantly, this manual fit method can only determine whether a part meets tolerance specifications, and it does not provide any information regarding how well the part meets tolerance specifications.

Similar to this manual fit, a geometric framework was developed to quantify the structure of positional tolerance evaluation [12]. A comparison between a genetic search method and a generalized reduced gradient method was done to explore methods for automatic analysis of inspection points for complex classes of objects [3].

In summary, so far there is a lack of an effective measure of part dimensional quality that is applicable to a variety of GD&T classes and conformal to ANSI Y14.5M standard. The *current practice using the minimum tolerance zone as a quality measure is computationally undesirable. Furthermore, it is not directly applicable to complex tolerances because the minimum tolerance zone characterizes part quality only through two surface envelops offset from one ideal geometry and complex tolerances often involve more than one tolerance feature. In this paper, part dimensional quality is quantified based on the amount of allowable transformation upon which a manufactured part shape remains within the tolerance zone. Such a measure is applicable to all GD&T classes where tolerance zones can be non-uniform, complex, or composite.*

ADMISSIBLE TRANSFORMATION VOLUME AND ITS PROPERTIES

The basic premise underpinning our approach is that the issue of part GD&T conformance check can be transformed into an issue of whether there exists a transformation such that, upon this transformation, the inspection points can be contained in the tolerance zone. Geometrically speaking, this is essentially a containment problem.

In this paper, we assume the inspection point set represents the actual manufacturing shape. So we use the term inspection point set and manufactured shape interchangeably in this paper. In addition, we do not consider measurement uncertainty.

PARAMETRIC TOLERANCE ZONE REPRESENTATION FOR PART QUALIFICATION

In order to conduct a containment check, an efficient representation of tolerance zone is needed. In this paper, we represent the tolerance zone as a distance function of nominal geometry. If the part surface has parametric representation s(u,v), we can then have tolerance zone represented as

$$Z = Z(u,v)$$

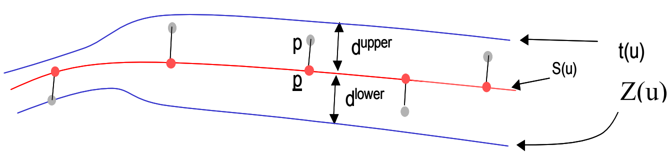


Figure 2: Parametric representation of tolerance zone

That is, given a surface point and its parameter set (u, v), we can calculate the tolerance band from the parameter set. Figure 2 illustrates a 2D example. For any given point p_i, we can find the closest point $\underline{p}_i(u_i, v_i)$ in the nominal surface. At this closest point, the tolerance can be represented as an interval $[d_i^l, d_i^u]$, which could be a symmetric two-sided tolerance, or asymmetric tolerance, or one-sided tolerance. We note the distance between the point p_i and the nominal geometry as a signed distance d

$$d = |p_i - \underline{p}_i| \cdot \tau$$

τ equals 1 if $\overrightarrow{\underline{p}_i p_i}$ has the same direction as part surface normal at point p. Otherwise, τ equals –1. So the point p_i lies within tolerance zone if and only if $d \in [d_i^l, d_i^u]$. The manufactured part meets tolerance specification if and only if all inspection points fall within the tolerance zone.

ADMISSIBLE TRANSFORMATION

If we represent tolerance zone as a set Z in an N-dimensional Euclidian space E^n, n=2, 3. Its boundary representation is described as a distance function from the nominal shape. Its coordinate system is represented as F_D, meaning a reference frame in design coordinate system.

The inspection point set is represented as *P* in the coordinate system F_I (a reference frame in inspection coordinate system). We assume if all the points in the point set P can be fit into the tolerance zone Z, the part is then conformal to part tolerance specifications. This can be formed mathematically as a containment problem as following: Given two sets P and Z, a part is conformal to tolerance specification if and only if under a transformation *t* such that P is contained within Z,

$$T(P,t) \subset Z$$

To better describe the process of computing such a transformation, we define a transformation space first.

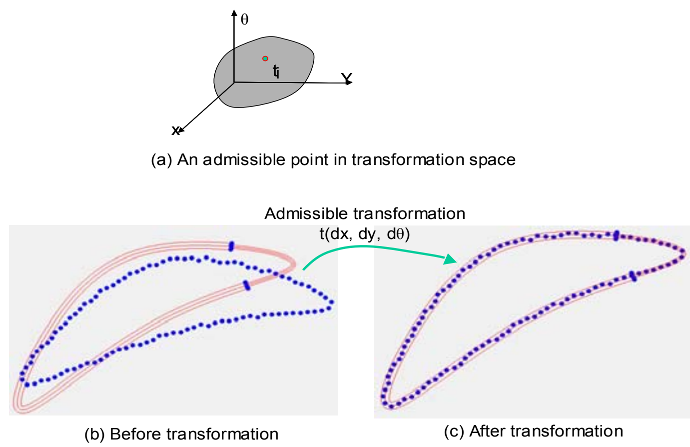


Figure 3: Transformation Space

For *n* degrees of freedom, we define an n-dimensional *transformation space* R^n, n=1,2,… 6. A more general representation of a point in the space could be (x,y,z,θ,φ,γ), respectively representing three translation components and three rotation components around x, y, and z axes. Each point in this transformation space represents a point t_i. A free rigid body has six degrees of freedom, three in translation and three in rotation. In the context of part GD&T conformance check, the degrees of freedom in fitting inspection data against nominal model/tolerance zone could be less than six [9]. For example, a minimum deviation zone of a straightness tolerance can be obtained by optimizing a one-parameter objective function. In the case of manual inspection of a surface profile through the optical comparator, there are three degrees of freedom for profile tolerance conformance check. They are two translations (x and y) and one rotation around the z-axis (θ). A point in such a transformation space represents a transformation (x_i, y_i, θ_i) applied to the measured point cloud *P* (Figure 3).

A point is an *admissible transformation point* if and only if such a transformation leads to the measured point set P falling with tolerance zone Z. That is, *t* is an admissible point if and only if

$$T(P,t) \subset Z .$$

A collection of all such admissible transformation points is called *admissible transformation volume (ATV)*. That is,

$$ATV=\{t\}$$

ATV PROPERTIES

As stated in [19], ATV has the following properties:

- A manufactured part is within tolerance specifications if and only if its admissible transformation volume is not null.
- The larger a tolerance is, the larger a part's admissible transformation volume is.
- The nominal design shape's admissible transformation volume ATV_D should be no smaller than the actual manufactured shape's admissible transformation volume ATV_M.

In this paper, we focus on two additional characteristics of ATV: transformation invariance and decomposability. The former enables robust ATV computing and the latter enables effective ATV computing for complex tolerance and also makes it possible to quantify each tolerance feature's influence over part conformance.

TRANSFORMATION INVARIANT

The shape, size and orientation of an ATV are invariant to the rigid body transformation of the coordinate data with reference to the tolerance zone.

This can be easily proved. If there exists an admissible transformation t_i for the point set P. We note the point set after the admissible transformation t_i as P_{t_i} and we have $P_{t_i} \subset Z$.

If an arbitrary rigid body transformation Δt is imposed on the point set P and we note the point set after transformation as $P_{\Delta t}$, we can obtain P from $P_{\Delta t}$ through an inverse transformation, i.e. $T(P_{\Delta t},-\Delta t) = P$. We know $T(P,t_i) \subset Z$. So we have $T(T(P_{\Delta t},-\Delta t),t_i) \subset Z$. That is the ATV for point set $P_{\Delta t}$ is $-\Delta t$ away from the ATV for the original point set P. Therefore, in the transformation space, the two ATVs are only off by a translation Δt.

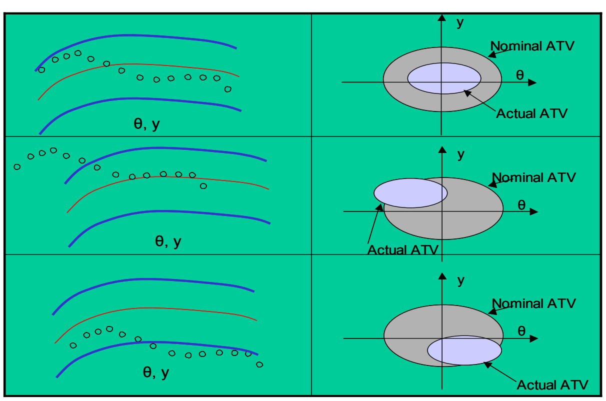


Figure 4: ATV property - transformation invariant

Figure 4 shows 1) different relative positions and orientations between the tolerance zone and point cloud, and 2) the corresponding ATVs with reference to nominal geometry's ATV. An ATV is determined by the surface points on the part boundary and the tolerance zone from design specifications. This implies that the initial position and orientation of coordinate data with refer to the tolerance zone will only affect the ATV's relative position in the transformation space, but will not affect the size, position and shape of the ATV.

DECOMPOSABILITY

An important property of an ATV is that the ATV of a complex part or of a part with composite tolerance can be computed through the intersection of ATVs of the decomposed tolerance features. The ATV of each decomposed tolerance feature is an ATV computed based on the measured points for the decomposed tolerance feature and the corresponding tolerance zone. This decomposable computing property is based on an important observation that, when a set of part surface points are decomposed into several subsets, the overall point set's ATV equals to the intersection of all the subsets' ATVs (EQ.1).

$$ATV = \cap_{i=1}^{n} ATV_i \qquad \textbf{EQ. 1}$$

The above property of ATV has two important implications: **enabling efficient ATV computing and easy identification of the producibility-limiting tolerance zone**.

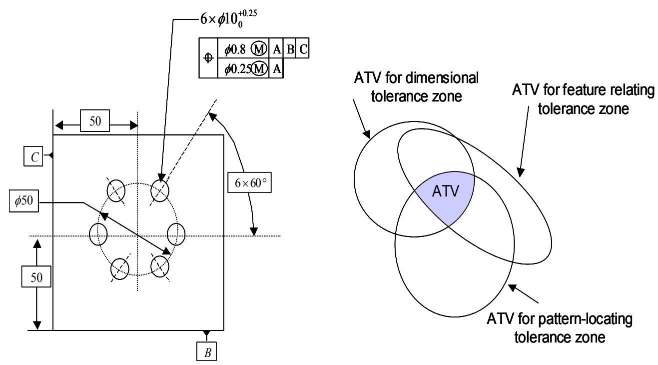


Figure 5: Decomposable ATV computing

Decomposed computing reduces computational complexity and saves the number of time the closest distance needs to be calculated. For a tolerance zone in the complex shaped geometry or composite positional tolerance such as patterned holes, the distance minimization in containment fit, least-squares fit, or minim-max fit methods can often lead to a local minimal solution and can be computationally inefficient. The distance function minimization is often done through the calculation of the closest distance between a coordinate point and the nominal geometry. For freeform curves or surfaces, the computing of such a shortest distance is an iterative process and consumes significant amount of time. Decomposable ATV computing can be utilized to compute ATVs for critical key points first and then to compute the ATV for entire geometry.

A second benefit of decomposable ATV computing is that the ATV enables the identification of the producibility-limiting tolerance zone for composite tolerance. In Figure 5 is a patterned-hole. There are three tolerance requirements for this part to be qualified: dimensional tolerance, feature relating tolerance and pattern-location tolerance. Neither least-squares nor mini-max fit is able to determine the composite tolerance conformance. Minimal tolerance zone computing is not directly applicable since three tolerance zones are involved. Thus far, the only way to determine the part conformance is through a containment test —— if all points are simultaneously within all three zones. However, this test does not provide quantitative measure of the part quality on whether it just fits in a tolerance zone or fits in with ample space remaining. **Decomposable ATV computing provides for the first time the ability to quantify, analyze and visualize the influence of each sub-tolerance-zone in a composite tolerance or sub-geometry in a complex shape over overall part conformance to GD&T specifications.**

A decomposable ATV computing computes ATVs for each sub-tolerance-zone separately. The intersection of ATVs then determines the overall part GD&T conformance. In the patterned hole example, the existence of three ATVs corresponding to three tolerance zones makes it transparent which of the three tolerance zones is more conformance limiting. Tolerance can then be re-allocated or the manufacturing process can be modified accordingly to improve part producibility.

One term that can be used to measure how much transformation allowance is actually utilized in overall GD&T conformance for each sub-tolerance zone is the volume ratio between the overall ATV and each individual ATV (EQ.2).

$$\mu = \frac{V_{ATV}}{V_{ATV_i}} \qquad \textbf{EQ. 2}$$

The ideal **ATV utilization ratio** (also the maximum ratio) is 1, meaning all the transformation allowance for each sub-tolerance-zone is fully utilized for the overall part GD&T conformance. As the ratio becomes smaller, more transformation allowance is not utilized due to other sub-tolerance-zones' restricting influence. When this happens, tolerance allocation and the manufacturing process need to be rectified to avoid such an uneconomical way of producing parts.

In order to conduct decomposed ATV computing, an **ATV intersection** algorithm can be developed to intersect sub-tolerance-features' ATVs into one ATV.

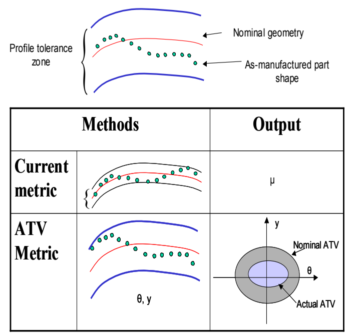
COMPARISON WITH MINIMAL TOLERANCE ZONE BASED METRIC

Minimal tolerance gives a theoretical minimum of the tolerance zone from envelopes of two ideal features with minimum separation distance within which the entire feature surface of the manufactured part must lie.

The ATV as a part dimensional quality metric relates to minimal tolerance zone in the following way:

- An ATV reflects the comparison between an actual part shape and design tolerance specification, while a minimum tolerance zone is a minimal zone bounding the actual tolerance feature regardless of design tolerance specification value.
- As minimal tolerance zone gets larger, ATV gets smaller. When the minimal tolerance zone of a part is smaller than design specified tolerance, there exists an ATV for this part. When the minimal tolerance zone is the same as design specified tolerance zone, the ATV is degenerated into a point. When the minimal tolerance zone is larger than the design tolerance specification, ATV is NULL and a volume enclosed by the hyper-iso-surface in the transformation space is used as a characterization of how bad the part quality is.
- Minimal tolerance zone is ineffective in handling simultaneous multiple tolerance zone specifications and multiple tolerance zone values may exist for one tolerance feature depending on the tolerance zone values for other tolerance features. ATV, as a single metric, is applicable to both single and multiple tolerance zone specifications.
- ATV enables decomposable computing to reduce computational complexity since it transforms the containment issue from the part geometry space into transformation space and thereby enabling set intersection operation for the composite ATV computing. Minimal tolerance zone, by its definition, needs to consider all the points simultaneously.

Figure 6 gives a comparison of part dimensional quality metric between minimal tolerance zone and ATV.



	Tolerance value	ATV
Applicability	Not applicable to composite tolerance features	Applicable to simple and composite tolerance features
Output representation	Single value	A hyper-volume with metric properties (length, volume)
Decomposable	Not decomposable	Decomposable
Computing	Dedicated algorithms, not computationally desirable	Unified algorithm

Figure 6: Dimensional quality metrics comparison

COMPUTING ATV

COMPUTE ADMISSIBLE POINT

In order to find an admissible point in the transformation space and to define the boundary of the admissible transformation volume, we define a distance function in the 3D nominal model's Euclidian Space. It is based on a containment fit function, the average distance between the points outside the tolerance zone and the tolerance zone boundary. That is, we only count the points outside the tolerance zone.

Mathematically, the objective function is defined as follows

$$\mathbf{f}=\sqrt[m]{\frac{\sum_{i=1}^{N}\left\|g(T(p_i,t)-s(u,v))\right\|^m}{N}} \qquad \textbf{EQ. 3}$$

$$g(T(p_i,t)-s)=\begin{cases}0 & \text{if } d\in[d_l,d_u]\\ |d-d_l| & \text{if } d<d_l\\ |d-d_u| & \text{if } d>d_u\end{cases} \qquad \textbf{EQ. 4}$$

In the above equations, t is transformation coordinates, $T(p_i,t)$ represents a transformation of point p_i by t. If t is a point in six-dimensional transformation space, t=(x, y, z, θ,φ,γ). N is the total number of points in the inspection point set. The symbol m represents the order of distance function g in the containment fit.

For any given set of parameters (in the transformation space), if the objective function is zero, this transformation point is an admissible point. If the minimal objective function value is not zero, the part is out of tolerance specification. The minimal value is an indication of how much the part is out of tolerance specification. The objective function f is a measure of the average distance between points outside of point boundary and the tolerance boundary.

If the objective function is larger than zero, it means under this transformation t_i, inspection points are still f distance away from lying within the tolerance zone. A collection of the transformation points that would lead to the same objective function value is called (hyper) iso-surface. If there are two degrees of freedom, such a collection would be an iso-curve. If there are three degrees of freedom, it would be an iso-surface. When there are more than three degrees of freedom, such a point set forms a hyper-iso-surface.

When the objective function value is zero, the corresponding iso-surface and the enclosed area in the transformation space form the ATV. We adopt a functional representation of the boundary of admissible transformation volume. That is,

$$f = \sqrt[m]{\frac{\sum_{i=1}^{N} \left\| g(T(p_i, t) - s(u,v)) \right\|^m}{N}} = 0 \qquad \textbf{EQ. 5}$$

This function describes all the admissible transformations.

COMPUTING ATV BOUNDARY POINT

An ATV boundary point refers to a point in the transformation space, upon which the coordinate data just touches the boundary.

The distance function in EQ.3 is useful for computing an admissible point. However, the function equals to zero for any point within the ATV. In order to compute the ATV boundary points, we introduce another set of distance computing functions: inside distance. We refer to the distance function in EQ.3 as *outside distance* and it is the sum of point distance outside tolerance boundary. *Inside distance* is the minimum distance between inside points and tolerance boundary. Mathematically, it can be represented as:

$$d^- = \begin{cases} \min\limits_{i=1,\dots N} (\|d_u - d_i\|, \|d_l - d_i\|) & \text{if } d_i \in [d^l, d^u] \\ 0 & \text{if } d_i \notin [d^l, d^u] \end{cases} \qquad \textbf{EQ. 6}$$

We note the combined inside and outside distance between the point set P and tolerance zone Z as $d(t)$.

$$d(t) = \begin{cases} d^+ & \text{if } d^+ > 0 \\ d^- & \text{if } d^+ == 0 \;\&\&\; d^- > 0 \\ 0 & \text{if } d^+ = d^- = 0 \end{cases} \qquad \textbf{EQ. 7}$$

Therefore, a transformation point t is an ATV boundary point if and only if $d(t)=0$.

An example outside distance function and combined distance function for the part in Figure 1.c is shown in Figure 7. As shown in Figure 7, the gradient of outside distance function in the ATV interior is zero, which makes it difficult to compute the ATV boundary points from an admissible point. On the other hand, the combined distance equals zero only when it is a boundary point. Therefore, the intersection point between a line in the transformation space and the ATV boundary happens if and only if $d(t)$ is zero, *i.e.* the inside distance is 0 and the outside distance is 0. The function $d(t)$ is continuous.

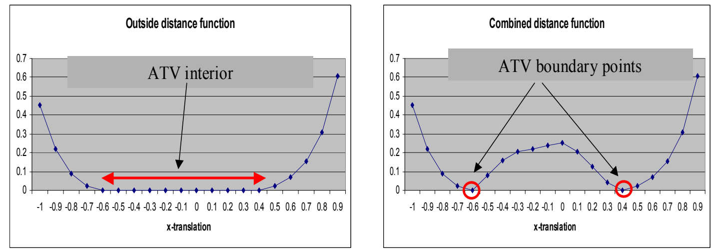


Figure 7: Distance functions for ATV boundary point computing

The combined distance function is at least C^0 continuous, which ensures ATV is differentiable for ATV sensitivity computing such as $\frac{\partial V_{ATV}}{\partial Z_i}$ and $\frac{\partial V_{ATV}}{\partial \sigma_j}$, in which V is the volume of the ATV, and Z_i and σ_j represents the i-th tolerance zone and j-th manufactured part dimensional deviation. These sensitivity measures can be used to characterize part producibility with reference to tolerance specification (Z_i) and manufacturing process variation (σ_j).

COMPUTING ATV

The combined outside/inside distance function is an implicit function defining the ATV boundary exactly. However, to compute its metric such as length and volume, numerical algorithms will need to be developed. One obvious approach for modeling such an implicit function is through space decomposition (such as voxel and octree representation) based approaches. A voxel representation in conjunction with Marching cubes was reported in [19]. An octree-based method is also implemented in this paper and shown in Figure 8.

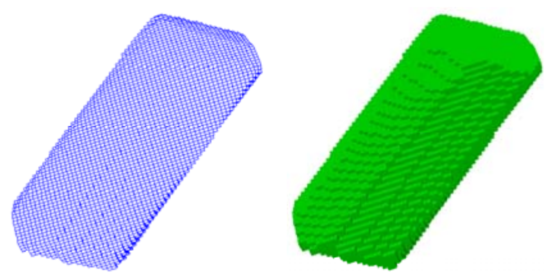


Figure 8: Octree representation of an ATV

In addition, in this paper, in order to support set-theoretic operation (EQ.3), we also construct the ATV from a collection of ATV boundary points. We then use a skinning operation to fit a solid volume to these boundary points. Specifically, it involves the following steps:

STEP 1: COMPUTE AN ADMISSIBLE POINT T_I

This admissible point can be computed from minimizing EQ.3 using a simplex optimization method. If the function value of

EQ.3 is zero, the t_i is an admissible point. If not, the ATV does not exist.
We use the centroid of the simplex in the transformation space as the admissible point for further ATV computing.

STEP 2: FIND THE ALLOWABLE TRANSFORMATION RANGE ALONG A DIRECTION

This can be done using the ATV boundary points computing method listed in Section 4.2. If we note the admissible point from STEP 1 as t_0 and the search direction of the allowable transformation as v, we can have a one-dimensional objective function:

$$h(\lambda) = d(t_0 + \lambda \cdot v) \qquad \textbf{EQ. 8}$$

Minimizing EQ.8 will lead to λ_1 for an ATV boundary point $t_0 + \lambda_1 \cdot v$. If we reverse the search direction in EQ.8, we will have another ATV boundary point as $t_0 - \lambda_2 \cdot v$. Therefore, the allowable transformation range along v would be $(\lambda_1 + \lambda_2) \cdot v$.

STEP 3: SAMPLE THE ATV CONTOURS AT DIFFERENT SECTION HEIGHT

We can choose one of the transformation axes, either a translation axis or a rotation axis, as a basis for sampling. For each sampled point, we can compute the ATV contours on a plane perpendicular to the transformation axis. This contour can be obtained using procedures described in STEP 1 and STEP2.

STEP 4: FORM THE SOLID THROUGH ALL THESE CONTOURS

A solid can be fit to pass through all the ATV contours. The examples of ATV computing are shown in the Experiment 2 in Section 5.

STEP 5: COMPUTING ATV THROUGH SET INTERSECTION

For complex tolerance check, we can compute ATV for each sub-tolerance feature and use set intersection operations to obtain the final resulting ATV as in EQ.1. Experiment 3 in Section 5 will demonstrate this.

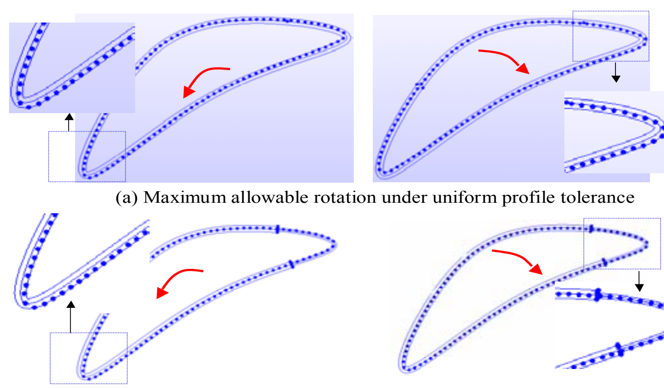
IMPLEMENTATION

To illustrate the basic concept and the efficacy of applying ATV for part dimensional quality gauging, we implemented a prototype system at Illinois Institute of Technology. The system is built on a Windows PC platform. The modeling kernel is *ACIS*® from Spatial Technology Inc.
Three sets of experiments were conducted on the profile tolerance example in Figure 1: i) *Computing the admissible transformation range along a particular direction*; ii) *Computing the ATV*; and iii) *Decomposed ATV computing for composite tolerance*. The cross-sectional shape is a closed degree three B-spline with control points as following (-50, -30,0), (-30,20,0), (-6,40,0), (50,30,0), (-4,20,0). The tolerances are 1.0 for the uniform profile tolerance and 1.0 and 0.5 for the non-uniform profile tolerance. Total 100 synthetic inspection points are created with deviation coefficient C=0 and C=0.5. Refer to [19] for detailed description of synthetic inspection point creation.

EXPERIMENT 1: *Computing the admissible transformation range along a particular direction*

For parts that are under single tolerance constraints, minimal tolerance zone can be a very effective means to characterize the dimensional quality. However, for parts under non-uniform tolerance constraints, admissible translation and rotation are often used as a simple and practical measure for part dimensional quality. For example, in the production shops, manual fitting of part profile against tolerance zone in a Mylar or plastic paper to characterize the fitting allowance is a common way of characterizing part quality under composite tolerance.

The admissible transformation range directly corresponds to the manual fitting result. Figure 9 shows the maximum allowable rotation for the part under uniform and non-uniform profile tolerances. Depending on the rotation direction, the inspection points touch the tolerance boundary at different locations. Using the method in STEP 2 in Section 4.3, we obtained the following results on the allowable transformation for the nominal cross section. Along both counter-clock-wise and clock-wise directions, the part under uniform profile tolerance can rotate 1.348 degrees without exceeding the tolerance boundary. When the part is under non-uniform profile tolerance, the allowable rotation angles are 0.674 degree. Similar computation can be done for x-translation and y-translation.

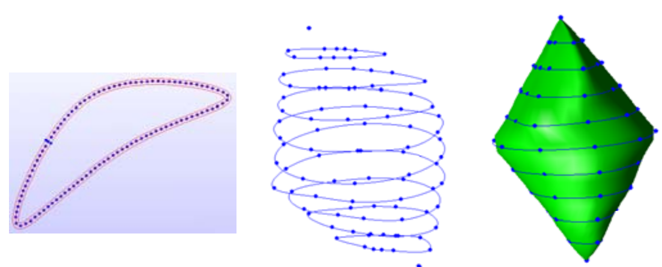


(a) Maximum allowable rotation under uniform profile tolerance

(b) Maximum allowable rotation under non-uniform profile tolerance

Figure 9: Maximum allowable rotation

EXPERIMENT 2: Computing the ATV



(a) Part in Euclidian space

(b) ATV boundary points

(c) ATV computed through skinning

Figure 10: ATV computing

To fully characterize part dimensional quality, an ATV will need to be computed. Figure 10 showed the ATV computing process for nominal part geometry under uniform profile tolerance. We chose the rotation direction as a sampling axis for computing ATV section contours. Ten boundary points per section with a total of eleven sections are used to construct the ATV. The volume of the ATV for the nominal geometry under uniform profile tolerance is 4.324. Volume for the part with deviation coefficient C=0.5 is 1.209 (Figure 11). The overlay of ATVs for nominal geometry and the actual geometry under uniform tolerance at different cross-sections is shown in Figure 12. This figure reveals that the nominal ATV encloses the actual ATV.

Figure 13 shows the nominal and actual part geometry and the overlay of the respective ATVs. The volumes of the ATV for nominal and actual geometry under non-uniform profile tolerances are 1.230 and 0.437.

(a) Part in Euclidian space (b) ATV in transformation space

Figure 11: ATV for an actual part

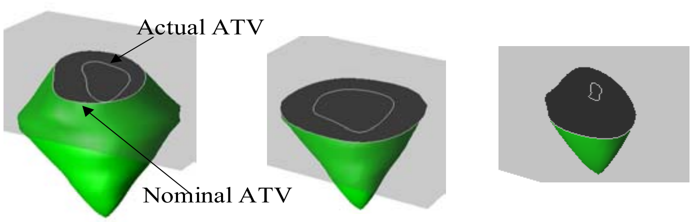
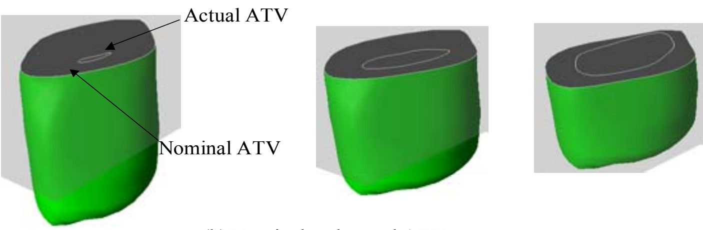


Figure 12: Overlay of an actual ATV and the nominal ATV

(a) Nominal and actual part geometry



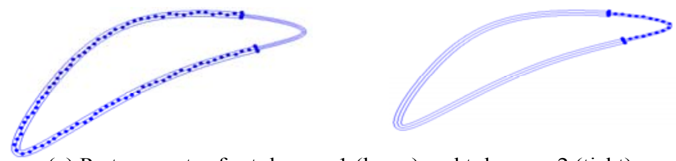
(b) Nominal and actual ATV

Figure 13: ATVs for non-uniform profile tolerance

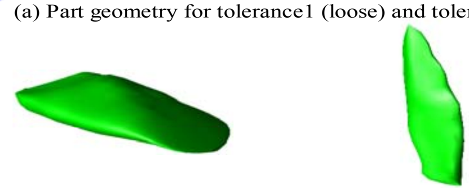
EXPERIMENT 3: *Decomposed ATV computing for composite tolerance*

The decomposable property of ATV enables an efficient way for ATV computation. Figure 14 shows an example of computing an ATV for the composite tolerance (non-uniform profile tolerance in this example) through a set intersection operation. The ATVs of part geometry under different tolerances are computed separately and they are subsequently intersected to produce the final resulting ATV. The volumes of the ATVs for the part geometry for tolerance 1 and tolerance 2 are 2.276 and 1.384. The volume for the intersected volume is 0.444. Therefore the ATV utilization ratios for each tolerance zone are 0.195 and 0.321. *Therefore, we can conclude the tolerance2 (tight) feature is more conformance-restrictive according to the utilization ratios*.

Figure 15 displays an overlay of the ATV computed from the set intersection and the ATV computed directly through procedures in Experiment 2. The contours at different sections are very close to each other. The slight discrepancy is attributed to the approximation in the skinning operation in ATV modeling since only limited ATV boundary points are used to the ATV construction.



(a) Part geometry for tolerance1 (loose) and tolerance 2 (tight)



(b) ATVs for tolerance1 (loose) and tolerance 2 (tight)

(c) ATV for composite tolerance through set intersection

Figure 14: ATV computing through set intersection

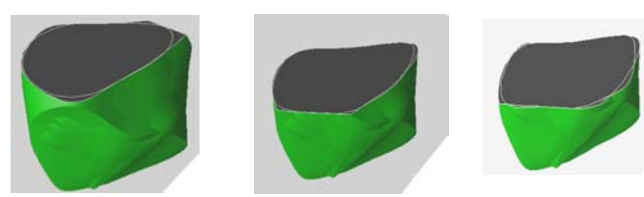


Figure 15: Overlay of ATVs computed from different methods

CONCLUSION

This paper presents a new metric, *admissible transformation volume*, for part dimensional quality gauging. It quantifies the dimensional quality through the amount of admissible transformation. Its efficacy includes its conformance to ANSI Y14.5M standard, broad applicability to both single tolerance and composite tolerance, its robustness due to the transformation invariant. Its decomposable computing property also enables the identification of the conformance-limiting part tolerance features.

Even though our mathematical formulation for ATV is directly applicable for tolerances in higher dimension, our algorithms implemented in this paper are limited to three degrees of freedom. Future work will focus on developing ATV computing algorithms in the higher-dimensional transformation space. Higher dimensional implicit function visualization methods such as scatter-plots, dimensional stacking, and parallel coordinates will be explored in the context of ATV visualization. Future work will also quantitatively examine the ATV based GD&T conformity in comparison with current approaches and its advantages in tolerance allocation.

REFERENCES

1. American National Standard ASME Y14.5M, *Dimensioning and Tolerancing*, The American Society of Mechanical Engineers, New York, 1994.

2. Barequet, G., Briggs, A.J., Dickerson, M.T., and Goodrich, M.T., "Offset-polygon annulus placement problems," *Computational Geometry: Theory and Applications* (CGTA), vol. 11 No. 3-4, pp. 125-141, 1998.

3. Carpinetti, L. C. R. and Chetwynd, D. G., "Genetic search methods for assessing geometric tolerances," *Computer Methods in Applied Mechanics and Engineering*, Vol. 122, pp. 193-204, 1995.

4. Choi, W. and Kurfess, T. R., "Dimensional Measurement Data Analysis, Part 1: A Zone Fitting Algorithm," ASME Transactions *Journal of Manufacturing Science and Engineering*, Vol. 121, pp. 238 – 245, 1999.

5. Choi, W. and Kurfess, T. R., "Dimensional Measurement Data Analysis, Part 2: Minimum Zone Evaluation," ASME Transactions *Journal of Manufacturing Science and Engineering*, Vol. 121, pp. 246 – 250, 1999.

6. Clement, A., Valade, C., Riviere, A., ''The TTRSs: 13 Oriented Constraints for Dimensioning, Tolerancing and Inspection,'' Advanced Mathematical Tools in Metrology III, Berlin, pp. 24–41, September 25–28,1996.

7. Damodarasamy, S., and Anand, S., ''Evaluation of Minimum Zone for Flatness by Normal Plane Method and Simplex Search,'' *IIE Transactions*, Vol. 31, pp. 617–626, 1999.

8. Davidson, J. K., Mujezinovic, A., and Shah, J. J., "A New Mathematical Model for Geometric Tolerances as Applied to Round Faces," ASME Transactions *Journal of Mechanical Design*, Vol. 124, pp. 609 – 622, 2002.

9. Endrias, D. H and Feng, H. Y., "Minimum-Zone Form Tolerance Evaluation Using Rigid-Body Coordinate Transformation," ASME Transactions *Journal of Computing and Information Science in Engineering*, Vol. 3, pp. 31 – 38, 2003

10. Feng, S. C. and Hopp, T. H., "A Review of Current Geometric Tolerancing Theories and Inspection Data Analysis Algorithms," NISTIR 4509, 1991.

11. Huang, S. T., Fan, K. C., and Wu, J. H., "A New Minimum Zone Method for Evaluating Straightness Errors," *Precision Engineering*, Vol. 15, pp. 158 – 165, 1993.

12. Kaiser, M. J., Cheraghi, S. H. and Li, S., "The structure of positional tolerance evaluation: I Constructive geometric approach," *Computer Methods in Applied Mechanics and Engineering*, Vol. 190, pp. 149-178, 2000.

13. Lai, H. Y., Jywe, W. Y., Chen, C. K., and Liu, C. H., ''Precision Modeling of Form Errors for Cylindricity Evaluation Using Genetic Algorithms,'' *Precision Engineering*, Vol. 24, pp. 310–319, 2000.

14. Lorensen, W. E. and Cline, H. E., "Marching Cubes: A High Resolution 3D Surface Construction Algorithm," *Computer Graphics* (Proceedings of SIGGRAPH '87), Vol. 21, No. 4, pp. 163-169, 1987.

15. Menq, C. H., Yao, H. T., and Lai, G. Y., "Automated Precision Measurement of Surface Profile in CAD-Directed Inspection," *IEEE Transactions on Robotics and Automation*, Vol. 8, No. 2, pp. 268-278, 1992.

16. Murthy, T. S. R., and Abdin, S. Z., ''Minimum Zone Evaluation of Surfaces,'' *International Journal of Machine Tools and Manufacture*, Vol. 20, pp. 123–136, 1980.

17. Pasupathy, T. M. K., Morse, E. P., and Wilhelm, R. G., "A Survey of Mathematical Methods for the Construction of Geometric Tolerance Zones," ASME Transactions *Journal of Computing and Information Science in Engineering*, Vol. 3, pp. 64 – 75, 2003.

18. Press, W. H., Teukolsky, S. A., Vetterling, W. T., and Flannery, B. P., *Numerical Recipes in C: The Art of Scientific Computin*g, 2nd ed., Cambridge University Press, New York, 1992.

19. Qian, X., Robinson, D. M., and Ross, J., "Admissible Transformation Volume for Part Dimensional Quality Gauging," *Computer-Aided Design*, in Press.

20. Requicha, A. A. G., "Towards a Theory of Geometric Tolerances," *Internal Journal of Robotics Research*, Vol. 2, No. 4, pp. 45-60, 1983.

21. Roy, U. and Li, B., 1998, ''Representation and Interpretation of Geometric Tolerances for Polyhedral Objects—I. Form Tolerances,'' *Computer-Aided Design*, Vol. 30, No.2, pp. 151–161, 1998.

22. Roy, U. and Zhang, X., ''Establishment of a Pair of Concentric Circles with the Minimum Radial Separation for Assessing Roundness Error,'' *Computer-Aided Design*, Vol. 24, No. 3, pp. 161–168, 1992.

23. Shah, J. J. and Zhang, B., "Attribute Graph Model for Geometric Tolerancing," Proceedings of 18th ASME Design Automation Conferences, Scottsdale, AZ, Sep 13-16, Vol. H0770B, pp. 133 – 139, 1992.

24. Turner, J. U. and Wozny, M. J., ''The M-space Theory of Tolerances,'' Proceedings of the ASME 16th Design Automation Conference, Chicago, IL, pp. 217–225, 1990.

25. Wu Y., Shah J., and Davidson J., "Computer Modeling of Geometric Variations in Mechanical Parts and Assemblies," ASME Transactions *Journal of Computing and Information Science in Engineering*, Vol. 3 No. 2, 2003.

